Perturbation Methods

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Algebraic equations

Many of the techniques of perturbation analysis can be introduced in the simple setting of algebraic equations. By starting with some particularly easy algebraic equations, three quadratics, we can benefit from the luxury of the existence of exact answers, taking useful hints from them to overcome difficulties.

1.1 Iteration and expansion

We start with the equation for x which contains the parameter ϵ,

$$x^2 + \epsilon x - 1 \;=\; 0$$

This has exact solutions

$$x \;=\; -\tfrac{1}{2}\epsilon \pm \sqrt{1 + \tfrac{1}{4}\epsilon^2}$$

which can be expanded for small ϵ as

$$x \quad = \quad \begin{cases} +1 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 - \frac{1}{128}\epsilon^4 + O(\epsilon^6) \\ -1 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{128}\epsilon^4 + O(\epsilon^6) \end{cases}$$

These binomial expansions converge if $|\epsilon| < 2$.

More important than converging, the truncated series give a good approximation if ϵ is small. The first few terms give a result within 3% of the exact result if

$$\begin{array}{rcccccccccc} |\epsilon| & < & 0.05 & & 0.5 & & 1.2 & & 1.6 & & \\ & & & \vdots & & \vdots & & \vdots & & \vdots & \\ x & = & 1 & - & \frac{1}{2}\epsilon & + & \frac{1}{8}\epsilon^2 & - & \frac{1}{128}\epsilon^4 & + & O(\epsilon^6) \end{array}$$

The last 1.6 being not too far from the convergence boundary. Alternatively for the fixed value of $\epsilon = 0.1$ the first few terms give

$$\begin{aligned} x &\sim 1.0 \\ &\quad 0.95 \\ &\quad 0.95125 \\ &\quad 0.95124921\ldots \\ &\quad \ldots \\ \text{exact} &= 0.95124922\ldots \end{aligned}$$

Often the numerical summation of these short expansions involves less computer time than the evaluation of the exact answer with its costly surds.

We started by finding the exact solution of the quadratic equation and then we expanded the exact solution. In most problems, however, it is not possible to find the exact solution. We must therefore develop techniques which first make the approximations and then, only afterwards, involve a calculation. There are two distinct methods of first approximating and then calculating, the iterative method and the expansion method. Each method has its own advantages and disadvantages.

Iterative method

We start with the iterative method, because it is a method which is often overlooked although it has much to offer.

The first step of the iterative method is to find a *rearrangement* of the original equation which will become the basis of an iterative process. This first step involves a certain amount of inspiration which must therefore count as a major drawback of the method. A suitable rearrangement of our present quadratic is

$$x = \pm\sqrt{1 - \epsilon x}$$

Any solution of the original equation is a solution of this rearrangement and vice versa.

Working with just the positive root, we thus adopt the iterative process

$$x_{n+1} = \sqrt{1 - \epsilon x_n}$$

The iterative process needs a starting point, the value of the root when $\epsilon = 0$, $x_0 = 1$.

Making the first iteration, we find

$$x_1 = \sqrt{1-\epsilon}$$

which can be expanded in a binomial series

$$x_1 = 1 - \tfrac{1}{2}\epsilon - \tfrac{1}{8}\epsilon^2 - \tfrac{1}{16}\epsilon^3 + \cdots$$

Looking at the exact answer, we see that the ϵ^2 and higher terms are erroneous. We therefore truncate the series for x_1 after the second term:

$$x_1 = 1 - \tfrac{1}{2}\epsilon + \cdots$$

Proceeding to the next iteration, we find

$$x_2 = \sqrt{1-\epsilon(1-\tfrac{1}{2}\epsilon)}$$

which can be expanded, this time retaining only terms up to ϵ^2:

$$\begin{aligned} x_2 &= 1 - \tfrac{1}{2}\epsilon(1-\tfrac{1}{2}\epsilon) - \tfrac{1}{8}\epsilon^2(1+\cdots)^2 + \cdots \\ &= 1 - \tfrac{1}{2}\epsilon + \tfrac{1}{8}\epsilon^2 + \cdots \end{aligned}$$

We note that the ϵ^2 term is now correct after two iterations. Iterating again, we find

$$\begin{aligned} x_3 &= \sqrt{1-\epsilon(1-\tfrac{1}{2}\epsilon+\tfrac{1}{8}\epsilon^2)} \\ &= 1-\tfrac{1}{2}\epsilon(1-\tfrac{1}{2}\epsilon+\tfrac{1}{8}\epsilon^2)-\tfrac{1}{8}\epsilon^2(1-\tfrac{1}{2}\epsilon+\cdots)^2-\tfrac{1}{16}\epsilon^3(1-\cdots)^3+\cdots \\ &= 1 - \tfrac{1}{2}\epsilon + \tfrac{1}{8}\epsilon^2 + 0\epsilon^3 + \cdots \end{aligned}$$

It is clear that progressively more work is required to obtain the higher order terms by the iterative method. The method also has the undesirable feature that in the early iterations it gives erroneous values to the higher terms. One can only check that a term is correct by making one more iteration, which of course is usually convincing but no rigorous proof (but see §1.5).

Expansion method

The first step of the expansion method is to set $\epsilon = 0$ and find the unperturbed roots $x = \pm 1$. Then one poses an expansion about one of these roots, say $x = +1$, expanding in powers of ϵ, i.e.

$$x(\epsilon) = 1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots$$

This expansion is formally substituted into the governing quadratic equation.

$$\begin{array}{rcccccccc} & 1 & + & \epsilon 2x_1 & + & \epsilon^2(x_1^2+2x_2) & + & \epsilon^3(2x_1x_2+2x_3) & +\cdots \\ & & + & \epsilon & + & \epsilon^2 x_1 & + & \epsilon^3 x_2 & +\cdots \\ & -1 & & & & & & & \\ = & 0 & & & & & & & \end{array}$$

Here one ignores potential difficulties in making the substitution such as the limitations in multiplying series term by term. The coefficients of the powers of ϵ on the two sides of the equation are now compared.

At ϵ^0: $\quad 1-1 = 0$

This level is satisfied automatically because we started the expansion from the correct value $x = 1$ at $\epsilon = 0$.

At ϵ^1: $\quad 2x_1 + 1 = 0 \qquad$ i.e. $x_1 = -\frac{1}{2}$

At ϵ^2: $\quad x_1^2 + 2x_2 + x_1 = 0 \qquad$ i.e. $x_2 = \frac{1}{8}$

Here the previously determined value of x_1 has been used.

At ϵ^3: $\quad 2x_1x_2 + 2x_3 + x_2 = 0 \qquad$ i.e. $x_3 = 0$

again using the previously determined values of x_1 and x_2.

The expansion method is much easier than the iterative method when working to higher orders. To use the expansion method, however, it is necessary to assume that the result can be expanded in powers of ϵ and that the formal substitution and associated manipulations are permitted.

Exercise 1.1. Find four terms in the expansion of the root near $x = -1$, using both the iterative and expansion methods.

1.2 Singular perturbations and rescaling

In this section we study the quadratic

$$\epsilon x^2 + x - 1 = 0$$

If $\epsilon = 0$ there is just one root, at $x = 1$, whereas when $\epsilon \neq 0$ there are two roots. This is an example of a *singular perturbation* problem, in which the limit point $\epsilon = 0$ differs in an important way from the approach to the limit $\epsilon \to 0$. Interesting problems are often singular. Problems which are not singular are said to be *regular*.

To resolve the paradox of the behaviour of the second root we take the exact solutions to the quadratic and expand them for small ϵ (convergent

if $|\epsilon| < \frac{1}{4}$). The two roots are

$$x = \begin{cases} 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + \cdots \\ -1/\epsilon - 1 + \epsilon - 2\epsilon^2 + 5\epsilon^3 + \cdots \end{cases}$$

Thus the singular second root evaporates off to $x = \infty$ in the limit $\epsilon = 0$.

Iterative method

To set up an iterative process for the singular root we argue as follows. In order to retain the second solution of the governing quadratic, it is necessary to keep the ϵx^2 term as a main term rather than as a small correction. Thus x must be large. Hence at leading order the -1 term in the equation will be negligible when compared with the x term, i.e.

$$\epsilon x^2 + x \approx 0 \qquad \text{with solution} \qquad x \sim -1/\epsilon$$

Hence we are led to the rearrangement of the quadratic

$$x = -\frac{1}{\epsilon} + \frac{1}{\epsilon x}$$

and the iterative process

$$x_{n+1} = -\frac{1}{\epsilon} + \frac{1}{\epsilon x_n}$$

with a starting point $x_0 = -1/\epsilon$.

Iterating once we find

$$x_1 = -\epsilon^{-1} - 1$$

and iterating again

$$\begin{aligned} x_2 &= -\epsilon^{-1} - \frac{1}{1+\epsilon} \\ &= -\epsilon^{-1} - 1 + \epsilon + \cdots \end{aligned}$$

A further iteration is needed to obtain the ϵ^2 term correctly.

Expansion method

The expansion method can be applied to the singular root by posing a power series in ϵ which starts with an ϵ^{-1} term instead of the usual ϵ^0. The way in which one determines the correct starting point is left until later in this section. Thus substituting

$$x(\epsilon) = \epsilon^{-1} x_{-1} + x_0 + \epsilon x_1 + \cdots$$

into the governing quadratic yields

$$\begin{array}{lllll} \epsilon^{-1}x_{-1}^2 & + & 2x_{-1}x_0 & + & \epsilon(2x_{-1}x_1 + x_0^2) + \cdots \\ \epsilon^{-1}x_{-1} & + & x_0 & + & \epsilon x_1 + \cdots \\ & - & 1 & & \\ & = & 0 & & \end{array}$$

Comparing coefficients of ϵ^n, we have

at ϵ^{-1}: $\quad x_{-1}^2 + x_{-1} = 0, \qquad$ i.e. $x_{-1} = -1$ or 0

The root $x_{-1} = 0$ leads to the regular root, so we ignore it.

At ϵ^0: $\quad 2x_{-1}x_0 + x_0 - 1 = 0, \qquad$ i.e. $x_0 = -1$

At ϵ^1: $\quad 2x_{-1}x_1 + x_0^2 + x_1 = 0, \qquad$ i.e. $x_1 = 1$

where at each stage the values of previously determined x_n have been used.

Rescaling in the expansion method

Instead of starting the expansion with the unusual ϵ^{-1} term, a very useful idea for singular problems is to rescale the variables before making the expansion. Thus introducing the rescaling

$$x = X/\epsilon$$

into the originally singular equation for x produces an equation for X,

$$X^2 + X - \epsilon = 0$$

which is regular. Thus the problem of finding the correct starting point for the expansion can be viewed as a problem of finding a suitable rescaling to regularise the singular problem.

There is a simple procedure to find all useful rescalings. First one poses a general rescaling with a scaling factor $\delta(\epsilon)$,

$$x = \delta X$$

in which one insists that X is strictly of order unity as $\epsilon \to 0$. Unfortunately the standard notation $X = O(1)$ does not describe this limitation on X, because $O(1)$ permits X to be vanishingly small as $\epsilon \to 0$. Thus we are forced to adopt the less familiar notation $X = \mathrm{ord}(1)$ to stand for X *is strictly of order unity as* $\epsilon \to 0$.

Substituting the general rescaling into the governing quadratic equation gives

$$\epsilon\delta^2 X^2 + \delta X - 1 = 0$$

We now consider the dominant balance of this equation for δ of different magnitudes, starting the search for sensible rescalings with δ very small and progressing to δ very large.

- $\delta \ll 1$. If δ is very small, then the left hand side of the equation is

$$\epsilon\delta^2 X^2 + \delta X - 1 \quad = \quad \text{small} + \text{small} - 1$$

This cannot balance the zero on the right hand side, and so a small δ is an unacceptable rescaling. As δ is increased, it is the X term which first breaks the domination of the 1 term and this occurs when $\delta = 1$. Hence the range of the unacceptable rescalings when δ is too small is $\delta \ll 1$, as declared above.

- $\delta = 1$. The left hand side of the equation is now

$$\epsilon\delta^2 X^2 + \delta X - 1 \quad = \quad \text{small} + X - 1$$

This can balance the zero on the right hand side to produce the regular root $X = 1 + \text{small}$.

- $1 \ll \delta \ll \epsilon^{-1}$. If δ is a little larger than unity, then the X term dominates the left hand side of the equation

$$(\epsilon\delta^2 X^2 + \delta X - 1)/\delta \quad = \quad \text{small} + X + \text{small}$$

This can balance the zero divided by δ on the right hand side, but only if $X = 0 + \text{small}$ which violates the restriction that X is strictly of order unity and not smaller. This rescaling is therefore unacceptable. As δ increases well beyond unity, it is the X^2 term which breaks the domination of the X term when $\delta = \epsilon^{-1}$. Hence this range of unacceptable rescalings is $1 \ll \delta \ll \epsilon^{-1}$, as declared above.

- $\delta = \epsilon^{-1}$. The left hand side of the equation divided by $\epsilon\delta^2$ is

$$(\epsilon\delta^2 X^2 + \delta X - 1)/\epsilon\delta^2 \quad = \quad X^2 + X + \text{small}$$

This can balance the zero divided by $\epsilon\delta^2$ on the right hand side with either $X = -1 + \text{small}$ which yields the singular root, or $X = 0 + \text{small}$ which is not permitted because it violates our restriction $X = \text{ord}(1)$.

- $\epsilon^{-1} \ll \delta$. Finally when δ is very large, the left hand side of the quadratic divided by $\epsilon\delta^2$ is

$$(\epsilon\delta^2 X^2 + \delta X - 1)/\epsilon\delta^2 \quad = \quad X^2 + \text{small} + \text{small}$$

This can only balance the right hand side if $X = 0 + \text{small}$, which violates $X = \text{ord}(1)$. Thus $\epsilon^{-1} \ll \delta$ is a range of unacceptable rescalings.

The systematic search of all possible rescalings has thus yielded $\delta = 1$ for the regular root and $\delta = \epsilon^{-1}$ for the singular root as the only possible rescalings with $X = \text{ord}(1)$.

Exercise 1.2. Find the rescalings for the roots of

$$\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0$$

and thence find two terms in the approximation for each root.

1.3 Non-integral powers

In this section we study the quadratic

$$(1-\epsilon)x^2 - 2x + 1 = 0$$

This innocent looking equation gives an unexpected surprise.

We start this time with the expansion method. Setting $\epsilon = 0$ we have the unperturbed solution $x = 1$, a double root. One learns from experience that a multiple root is a sign of imminent danger. Proceeding however in the usual manner, we pose the expansion in powers of ϵ

$$x(\epsilon) = 1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

Substituting into the governing equation

$$\begin{array}{rcccl} 1 & + & \epsilon 2x_1 & + & \epsilon^2(2x_2 + x_1^2) \quad + \cdots \\ & - & \epsilon & - & \epsilon^2 2x_1 \qquad\quad + \cdots \\ -2 & - & \epsilon 2x_1 & - & \epsilon^2 2x_2 \qquad\quad + \cdots \\ +1 & & & & \\ = 0 & & & & \end{array}$$

and comparing coefficients of ϵ^n we find

At ϵ^0: $\quad 1 - 2 + 1 = 0$

which is automatically satisfied because we started correctly perturbing about $x = 1$.

At ϵ^1: $\quad 2x_1 - 1 - 2x_1 = 0$

This cannot be satisfied with any value of x_1, except in some sense with $x_1 = \infty$.

To find the cause of the difficulty we look at the exact solution of the quadratic

$$x = \frac{1 \pm \epsilon^{\frac{1}{2}}}{1-\epsilon}$$

Taking just the positive root and expanding for small ϵ, we find

$$x = 1 + \epsilon^{\frac{1}{2}} + \epsilon + \epsilon^{\frac{3}{2}} + \cdots$$

Thus we see that we should have expanded in powers of $\epsilon^{\frac{1}{2}}$ rather than powers of ϵ. This is what the infinite value that we found above for x_1 was hinting: in a certain sense $\epsilon^{\frac{1}{2}} = \epsilon \times \infty$. In retrospect we could also have foreseen that an $O(\epsilon^{\frac{1}{2}})$ change in the variable would be required to produce an $O(\epsilon)$ change in a function at its minimum. Returning to the quadratic, we now pose an expansion in powers of the unexpected non-integral powers

$$x(\epsilon) = 1 + \epsilon^{\frac{1}{2}} x_{\frac{1}{2}} + \epsilon x_1 + \epsilon^{\frac{3}{2}} x_{\frac{3}{2}} + \cdots$$

Substituting this into the governing equation

$$\begin{array}{rcccccccc} 1 & + & \epsilon^{\frac{1}{2}} 2x_{\frac{1}{2}} & + & \epsilon(2x_1 + x_{\frac{1}{2}}^2) & + & \epsilon^{\frac{3}{2}}(2x_{\frac{3}{2}} + 2x_{\frac{1}{2}} x_1) & + \cdots \\ & & & - & \epsilon & - & \epsilon^{\frac{3}{2}} 2x_{\frac{1}{2}} & + \cdots \\ -2 & - & \epsilon^{\frac{1}{2}} 2x_{\frac{1}{2}} & - & \epsilon 2x_1 & - & \epsilon^{\frac{3}{2}} 2x_{\frac{3}{2}} & + \cdots \\ +1 & & & & & & & \\ = 0 & & & & & & & \end{array}$$

Comparing coefficients of $\epsilon^{\frac{n}{2}}$ we find that as usual

at ϵ^0: $\quad 1 - 2 + 1 = 0$

is automatically satisfied and that

at $\epsilon^{\frac{1}{2}}$: $\quad 2x_{\frac{1}{2}} - 2x_{\frac{1}{2}} = 0$

This is satisfied by all values of $x_{\frac{1}{2}}$. It is a little disturbing that $x_{\frac{1}{2}}$ has not been determined at the $\epsilon^{\frac{1}{2}}$ level, but we proceed to the next level.

At ϵ^1: $\quad 2x_1 + x_{\frac{1}{2}}^2 - 1 - 2x_1 = 0$

So $x_{\frac{1}{2}} = \pm 1$ and x_1 is not determined at this level. Continuing to the next level

at $\epsilon^{\frac{3}{2}}$: $\quad 2x_{\frac{3}{2}} + 2x_{\frac{1}{2}} x_1 - 2x_{\frac{1}{2}} - 2x_{\frac{3}{2}} = 0$

So $x_1 = 1$ for both roots of $x_{\frac{1}{2}}$, while $x_{\frac{3}{2}}$ is not determined.

The delay in determining $x_{\frac{n}{2}}$ at the $\epsilon^{\frac{n+1}{2}}$ level rather than at the $\epsilon^{\frac{n}{2}}$ level means that a little extra work is required. There is also the slight worry that at the following level it will not be possible to satisfy the equation and the whole solution will therefore collapse, as happened at the ϵ level in the erroneous expansion in powers of ϵ.

Finding the expansion sequence

Having rescued the expansion method by looking at the exact answer, there remains the problem of how one determines the expansion sequence when the exact answer is not available.

First one poses a general expansion

$$x(\epsilon) \;=\; 1 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2 + \cdots$$

where one requires

$$1 \gg \delta_1(\epsilon) \gg \delta_2(\epsilon) \gg \ldots \quad \text{and} \quad x_1, x_2, \ldots = \text{ord}(1) \text{ as } \epsilon \to 0$$

Substituting into the governing quadratic yields

$$\begin{aligned} & 1 + 2\delta_1 x_1 + \delta_1^2 x_1^2 + 2\delta_2 x_2 + 2\delta_1\delta_2 x_1 x_2 + \delta_2^2 x_2^2 + \cdots \\ & \quad - \epsilon - 2\epsilon\delta_1 x_1 - \epsilon\delta_1^2 x_1^2 - 2\epsilon\delta_2 x_2 + \cdots \\ & -2 - 2\delta_1 x_1 - 2\delta_2 x_2 + \cdots \\ & +1 \\ & = 0 \end{aligned}$$

While the relative magnitude of some of the terms is clear, e.g. $2\delta_1 x_1 \gg \delta_1^2 x_1^2$ and $2\delta_1 x_1 \gg 2\delta_2 x_2$ because $1 \gg \delta_1$ and $\delta_1 \gg \delta_2$ respectively, there is considerable uncertainty about the ordering of other terms, e.g. between $\delta_1^2 x_1^2$ and $2\delta_2 x_2$. Removing the cancelling terms one is left with

$$\begin{aligned} \delta_1^2 x_1^2 + 2\delta_1\delta_2 x_1 x_2 + \delta_2^2 x_2^2 + \cdots \\ -\epsilon - 2\epsilon\delta_1 x_1 - \epsilon\delta_1^2 x_1^2 - 2\epsilon\delta_2 x_2 + \cdots \;=\; 0 \end{aligned}$$

Using $1 \gg \delta_1 \gg \delta_2$ one can see that the leading order terms from the two lines are $\delta_1^2 x_1^2$ and $-\epsilon$. Therefore there are three possible leading order balances:

$$\begin{array}{llll} \text{either} & \delta_1^2 x_1^2 = 0 & \text{if} & \delta_1^2 \gg \epsilon \\ \text{or} & \delta_1^2 x_1^2 - \epsilon = 0 & \text{if} & \delta_1^2 = \epsilon \\ \text{or} & -\epsilon = 0 & \text{if} & \delta_1^2 \ll \epsilon \end{array}$$

Clearly the last option is unacceptable and so too is the first because we require $x_1 = \text{ord}(1)$. Hence we conclude that

$$\delta_1 = \epsilon^{\frac{1}{2}} \quad \text{and} \quad x_1 = \pm 1$$

Removing these two balancing terms leaves as leading order terms $2\delta_1\delta_2 x_1 x_2$ and $-2\epsilon\delta_1 x_1$. Repeating the above arguments

$$\begin{array}{llll} \text{either} & 2\epsilon^{\frac{1}{2}}\delta_2 x_1 x_2 = 0 & \text{if} & \delta_2 \gg \epsilon \\ \text{or} & 2\epsilon^{\frac{1}{2}}\delta_2 x_1 x_2 - 2\epsilon^{\frac{3}{2}} x_1 = 0 & \text{if} & \delta_2 = \epsilon \\ \text{or} & -2\epsilon^{\frac{3}{2}} x_1 = 0 & \text{if} & \delta_2 \ll \epsilon \end{array}$$

The only acceptable option is

$$\delta_2 = \epsilon \quad \text{and} \quad x_2 = 1 \quad \text{(for both } x_1 \text{ roots)}$$

Because the above determination of the expansion sequence involves some messy intermediate details, in practice one would take two attempts at the problem to determine δ_1 and δ_2. First one would substitute $x = 1 + \delta_1 x_1$ and find $\delta_1 = \epsilon^{\frac{1}{2}}$. Then one would substitute $x = 1 + \epsilon^{\frac{1}{2}} x_1 + \delta_2 x_2$ and find $\delta_2 = \epsilon$. Splitting the problem up into stages, one has to consider at each stage less terms of undetermined magnitude.

Iterative method

Finally the superiority of the iterative method should be noted in cases where the expansion sequence is not known. A suitable rearrangement of the original quadratic is

$$(x-1)^2 = \epsilon x^2$$

which leads to the iterative process

$$x_{n+1} = 1 \pm \epsilon^{\frac{1}{2}} x_n$$

Starting with $x_0 = 1$, the positive root gives

$$x_1 = 1 + \epsilon^{\frac{1}{2}}$$

and

$$x_2 = 1 + \epsilon^{\frac{1}{2}} + \epsilon$$

Not only is this considerably quicker but there is also no awkward step like the $\epsilon^{\frac{1}{2}}$ level in the expansion method which leaves $x_{\frac{1}{2}}$ undetermined.

Exercise 1.3. Find the first two terms of $x(\epsilon)$ the solution near 0 of

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - 1 - x + \tfrac{1}{2}x^2 = -\tfrac{1}{6}\epsilon$$

Exercise 1.4. Find the first two terms for all four roots of

$$\epsilon x^4 - x^2 - x + 2 = 0$$

Exercise 1.5 (Stone). Find the first two terms for all three roots of

a: $\epsilon x^3 + x^2 + (2+\epsilon)x + 1 = 0$

b: $\epsilon x^3 + x^2 + (2-\epsilon)x + 1 = 0$

1.4 Logarithms

In this section we shall find the solution as $\epsilon \to 0$ (through positive values) of the transcendental equation

$$xe^{-x} = \epsilon$$

One root is near $x = \epsilon$ which is easy to obtain. The other root becomes large as $\epsilon \to 0$ and is more difficult to find. We concentrate on this large root. As the expansion sequence is distinctly unclear, we employ the iterative method.

First we observe that if ϵ is small ($\epsilon < \frac{1}{4}$ is sufficient) the root must lie between $x = \ln(1/\epsilon)$ (for which $xe^{-x} = \epsilon \ln(1/\epsilon) > \epsilon$) and $x = 2\ln(1/\epsilon)$ (for which $xe^{-x} = \epsilon^2 2\ln(1/\epsilon) < \epsilon$). Over this range of x, the x factor merely doubles while the e^{-x} factor falls by an order of magnitude from ϵ to ϵ^2. Thus we can view the x factor as varying weakly and concentrate on the rapid variation in the e^{-x} factor. This suggests the rearrangement of the original equation

$$e^{-x} = \frac{\epsilon}{x}$$

leading to the iterative scheme

$$x_{n+1} = \ln(1/\epsilon) + \ln x_n$$

Further, from the above observations it is clear that the root must lie quite near $\ln(1/\epsilon)$ when ϵ is small. Thus we start the iteration from

$$x_0 = \ln(1/\epsilon)$$

Then

$$x_1 = \ln(1/\epsilon) + \ln\ln(1/\epsilon) = L_1 + L_2$$

where we have introduced the shorthand notation

$$L_1 = \ln(1/\epsilon) \quad \text{and} \quad L_2 = \ln\ln(1/\epsilon)$$

Iterating again

$$\begin{aligned} x_2 &= L_1 + \ln\left[L_1\left(1 + \frac{L_2}{L_1}\right)\right] \\ &= L_1 + L_2 + \frac{L_2}{L_1} - \frac{L_2^2}{2L_1^2} + \cdots \end{aligned}$$

And again

$$x_3 = L_1 + \ln\left[L_1\left(1 + \frac{L_2}{L_1} + \frac{L_2}{L_1^2} - \frac{L_2^2}{2L_1^3}\right)\right]$$

$$
\begin{aligned}
&= L_1 + L_2 + \left(\frac{L_2}{L_1} + \frac{L_2}{L_1^2} - \frac{L_2^2}{2L_1^3}\right) \\
&\quad - \tfrac{1}{2}\left(\frac{L_2}{L_1} + \frac{L_2}{L_1^2} + \cdots\right)^2 + \tfrac{1}{3}\left(\frac{L_2}{L_1} + \cdots\right)^3 + \cdots \\
&= L_1 + L_2 + \frac{L_2}{L_1} + \frac{-\frac{1}{2}L_2^2 + L_2}{L_1^2} + \frac{\frac{1}{3}L_2^3 - \frac{3}{2}L_2^2 + \cdots}{L_1^3} + \cdots
\end{aligned}
$$

The expansion sequence needed by the expansion method is clearly a tough one to guess. Moreover the iterative method produces more than one extra term from each iteration.

The appearance of $\ln\ln(1/\epsilon)$ means that remarkably small values of ϵ are required to achieve a good numerical accuracy of the approximate expressions. Usually one hopes for a tolerable agreement with $\epsilon < 0.5$ or at worse $\epsilon < 0.1$. In order for $\ln\ln(1/\epsilon) > 3$ however one needs $\epsilon < 10^{-9}$. The table below gives the percentage errors at various ϵ for the first five approximations to the large root of our transcendental equation.

ϵ	L_1	$+L_2$	$+L_2/L_1$	$-\frac{1}{2}L_2^2/L_1^2$	$+L_2/L_1^2$
10^{-1}	36	12	2	4	0.03
10^{-3}	24	3	0.02	0.04	0.04
10^{-5}	19	1	0.04	0.1	0.001

The table shows that acceptable accuracy is only achieved with many terms of the approximation or with extremely small values of ϵ. The table also demonstrates another common feature of expansions which involve $\ln\ln(1/\epsilon)$. This is that it is unwise to split $(-\frac{1}{2}L_2^2 + L_2)/L_1^2$ into two terms, because the error is made worse by the first part before it is eventually improved by the addition of the second part (at least at values of ϵ not astronomically small).

Exercise 1.6. Find several terms in an approximation for the solution of

$$\frac{e^{-x^2}}{x} = \epsilon$$

1.5 Convergence

The expansion method offers little opportunity of proving that an approximation converges. In straightforward problems the form of the n^{th} term will be clear, e.g. ϵ^n, and so one can be satisfied that the expansion is consistent. Just occasionally one can write down the problem for the general n^{th} term, find a strong bound on the magnitude of the term, and thence prove convergence of the expansion. In more difficult problems, however, the expansion sequence will not be clear and one would have no idea of the form of the general term. In these problems one can only be satisfied that the expansion is consistent as far as one has proceeded.

The iterative method on the other hand provides a simple proof of convergence. Suppose $x = x_*$ is the root of the equation

$$x = f(x)$$

where f is used in an iterative process $x_{n+1} = f(x_n)$. Then one iteration will take $x = x_* + \delta$ to

$$f(x_* + \delta) = x_* + \delta f'(x_*) + o(\delta)$$

if δ is small. Thus one iteration will decrease the error if

$$|f'(x_*)| < 1$$

Hence by the contraction mapping theorem, the iterative process will converge onto the root x_* if $|f'(x_*)| < 1$ and if the iteration is started sufficiently near to the root. (The standard theorem needs a small modification to take account of the truncation of the higher order terms which are known to be incorrect after insufficient iterations.)

In the previous sections we had iterative schemes which converge.

In §1.1	$f = \sqrt{1-\epsilon x}$	with $x_* \sim 1$	so $f'(x_*) \sim -\frac{1}{2}\epsilon$
In §1.2	$f = -1/\epsilon + 1/\epsilon x$	with $x_* \sim -1/\epsilon$	so $f'(x_*) \sim -\epsilon$
In §1.3	$f = 1 + \epsilon^{1/2} x$	with $x_* \sim 1$	so $f'(x_*) \sim \epsilon^{1/2}$
In §1.4	$f = \ln(\frac{1}{\epsilon}) + \ln(x)$	with $x_* \sim \ln(\frac{1}{\epsilon})$	so $f'(x_*) \sim 1/\ln(\frac{1}{\epsilon})$

The negative sign of f' in the first two cases means that the error changes sign and so two successive iterations must bracket the answer. Also from the magnitude of f' one can work out how many terms will be correct after a given number of iterations.

1.6 Eigenvalue problems

In this section we consider the eigenvalue problem for the eigenvalue λ associated with the eigenvector $\mathbf{x}$

$$A\mathbf{x} + \epsilon\mathbf{B}(\mathbf{x}) = \lambda\mathbf{x}$$

In order for this to qualify as an algebraic equation, A ought to be a matrix. The techniques of this section, however, can be applied to any linear operator A with adequate compactness. As $\epsilon\mathbf{B}(\mathbf{x})$ is a small perturbation, there is no need for $\mathbf{B}(\mathbf{x})$ to be linear.

We look for the perturbed eigensolution near to a given unperturbed eigensolution with eigenvalue a and associated eigenvector $\mathbf{e}$:

$$A\mathbf{e} = a\mathbf{e}$$

If the matrix A is not symmetric, its transpose will have a different eigenvector $\mathbf{e}^\dagger$ associated with the same eigenvalue:

$$\mathbf{e}^\dagger A = a\mathbf{e}^\dagger$$

Initially we restrict attention to the case where a is a single root with only one independent eigenvector $\mathbf{e}$. Then $\mathbf{e}^\dagger$ is orthogonal to all the other eigenvectors of A.

In the standard way we pose an expansion in powers of ϵ starting from the unperturbed eigensolution

$$\begin{aligned} \mathbf{x}(\epsilon) &= \mathbf{e} + \epsilon\mathbf{x}_1 + \epsilon^2\mathbf{x}_2 + \cdots \\ \lambda(\epsilon) &= a + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots \end{aligned}$$

Substituting into the governing equation and comparing coefficients of ϵ^n produces

at ϵ^0: $A\mathbf{e} = a\mathbf{e}$ which is automatically satisfied

at ϵ^1: $A\mathbf{x}_1 + \mathbf{B}(\mathbf{e}) = a\mathbf{x}_1 + \lambda_1\mathbf{e}$

It is useful to rearrange the last equation as

$$(A - a)\mathbf{x}_1 = \lambda_1\mathbf{e} - \mathbf{B}(\mathbf{e})$$

Now the left hand side of this equation can have no component in the direction of $\mathbf{e}$, because for all $\mathbf{x}_1$

$$\mathbf{e}^\dagger \cdot [(A - a)\mathbf{x}_1] = [\mathbf{e}^\dagger(A - a)] \cdot \mathbf{x}_1 = (a - a)\mathbf{e}^\dagger \cdot \mathbf{x}_1 = 0$$

using the eigenvector property of $\mathbf{e}^\dagger$. Thus there can exist no solution of the equation for $\mathbf{x}_1$ unless the right hand side of the equation also has

no component in the direction of $\mathbf{e}$, i.e.

$$\mathbf{e}^\dagger \cdot [\lambda_1 \mathbf{e} - \mathbf{B}(\mathbf{e})] = 0$$

Thus we have found the first perturbation of the eigenvalue

$$\lambda_1 = \frac{\mathbf{e}^\dagger \cdot \mathbf{B}(\mathbf{e})}{\mathbf{e}^\dagger \cdot \mathbf{e}}$$

Note that this expression shows that if $\mathbf{B}(\mathbf{e})$ is nonlinear then the eigenvalue is not independent of the magnitude of the eigenvector.

We now can return to the equation for $\mathbf{x}_1$ and substitute the expression for λ_1 to yield

$$(A-a)\mathbf{x}_1 = -\mathbf{B}(\mathbf{e}) + \frac{\mathbf{e}^\dagger \cdot \mathbf{B}(\mathbf{e})}{\mathbf{e}^\dagger \cdot \mathbf{e}}\mathbf{e} = -\mathbf{B}(\mathbf{e})_\perp$$

where the notation $\mathbf{B}(\mathbf{e})_\perp$ has been introduced for that part of $\mathbf{B}(\mathbf{e})$ perpendicular to $\mathbf{e}$. Now that the right hand side has no component in the direction of $\mathbf{e}$ it is possible to invert $(A-a)$ to obtain a solution for $\mathbf{x}_1$, although this solution is not unique because it is possible to add an arbitrary multiple of $\mathbf{e}$ to $\mathbf{x}_1$ without changing $(A-a)\mathbf{x}_1$. Thus with k_1 an arbitrary scalar

$$\mathbf{x}_1 = -(A-a)^{-1}\mathbf{B}(\mathbf{e})_\perp + k_1\mathbf{e}$$

where the restricted inverse $(A-a)^{-1}$ does exist in the space orthogonal to $\mathbf{e}$. If the complete eigendecomposition of A is known, then $\mathbf{x}_1$ can be represented as a sum over all the other eigenvectors $\mathbf{e}^{(j)}$

$$\mathbf{x}_1 = \sum_j{}' \frac{\mathbf{e}^{(j)\dagger} \cdot \mathbf{B}(\mathbf{e})}{(a-a^{(j)})(\mathbf{e}^{(j)\dagger} \cdot \mathbf{e}^{(j)})}\mathbf{e}^{(j)} + k_1\mathbf{e}$$

This completes the first order perturbation.

Second order perturbation

The second order perturbation is governed by

$$A\mathbf{x}_2 + \mathbf{B}_1 = a\mathbf{x}_2 + \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{e}$$

Here $\epsilon\mathbf{B}_1$ is the ord(ϵ) change from $\mathbf{B}(\mathbf{e})$ to $\mathbf{B}(\mathbf{e}+\epsilon\mathbf{x}_1)$. If $\mathbf{B}$ is linear, then $\mathbf{B}_1 = B\mathbf{x}_1$. If $\mathbf{B}$ is nonlinear, then $\mathbf{B}_1 = \mathbf{x}_1 \cdot \mathbf{B}'(\mathbf{e})$ where $\mathbf{B}'$ is the first derivative of $\mathbf{B}$. Rearranging the equation for $\mathbf{x}_2$ we have

$$(A-a)\mathbf{x}_2 = \lambda_2\mathbf{e} + \lambda_1\mathbf{x}_1 - \mathbf{B}_1$$

As in the problem for $\mathbf{x}_1$, we must require that the right hand side has no component in the direction of $\mathbf{e}$. This leads to the second perturbation

in the eigenvalue

$$\lambda_2 \;=\; \frac{\mathbf{e}^\dagger \cdot (\mathbf{B}_1 - \lambda_1 \mathbf{x}_1)}{\mathbf{e}^\dagger \cdot \mathbf{e}}$$

and thence, with k_2 an arbitrary scalar, the second perturbation in the eigenvector is

$$\mathbf{x}_2 \;=\; -(A-a)^{-1}(\mathbf{B}_1 - \lambda_1 \mathbf{x}_1)_\perp + k_2 \mathbf{e}$$

We see in the expressions for λ_2 and $\mathbf{x}_2$ that it would have been convenient to remove the non-uniqueness in $\mathbf{x}_1$ by requiring it to have no component in the direction of the unperturbed eigenvector, i.e. $\mathbf{e}^\dagger \cdot \mathbf{x}_1 = 0$. In some problems, however, there are more pressing claims than this convenient normalisation.

If the complete eigendecomposition of A is known, and also if $\mathbf{B}$ is linear, then our result for λ_2 can be written in the more familiar form

$$\lambda_2 \;=\; {\sum_j}' \frac{(\mathbf{e}^\dagger \cdot B\mathbf{e}^{(j)})(\mathbf{e}^{(j)\dagger} \cdot B\mathbf{e})}{(a-a^{(j)})(\mathbf{e}^{(j)\dagger} \cdot \mathbf{e}^{(j)})(\mathbf{e}^\dagger \cdot \mathbf{e})}$$

Multiple roots

Suppose that the eigenvalue a of A is associated with more than one independent, non-degenerate eigenvector, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. We must now consider perturbing around a general eigenvector in the eigenspace

$$\begin{array}{rcccl} \mathbf{x} & = & \sum_{i=1}^n \alpha_i \mathbf{e}_i & + \epsilon \mathbf{x}_1 & + \cdots \\ \lambda & = & a & + \epsilon \lambda_1 & + \cdots \end{array}$$

Substituting into the governing equation and comparing coefficients of ϵ^n produces at ϵ^1

$$(A-a)\mathbf{x}_1 \;=\; \lambda_1 \textstyle\sum_{i=1}^n \alpha_i \mathbf{e}_i \;-\; \mathbf{B}\left(\sum_{i=1}^n \alpha_i \mathbf{e}_i\right)$$

In this case of multiple roots, the left hand side can have no component in the eigenspace. Thus requiring the right hand side to have no component in each of the independent directions $\mathbf{e}_i$ produces

$$\begin{array}{rcl} \lambda_1 \alpha_1 & = & \mathbf{e}_1^\dagger \cdot \mathbf{B}\left(\sum_i \alpha_i \mathbf{e}_i\right) / (\mathbf{e}_1^\dagger \cdot \mathbf{e}_1) \\ \vdots & & \vdots \\ \lambda_1 \alpha_n & = & \mathbf{e}_n^\dagger \cdot \mathbf{B}\left(\sum_i \alpha_i \mathbf{e}_i\right) / (\mathbf{e}_n^\dagger \cdot \mathbf{e}_n) \end{array}$$

These equations are a new eigenvalue problem in the eigenspace of A to find the eigenvalue λ_1 and eigenvector α. If $\mathbf{B}$ is linear there will exist n eigenvalues and, except in some degenerate cases, n associated independent eigenvectors. If $\mathbf{B}$ is nonlinear, it is possible that no eigensolutions

exist. In such cases the original eigenproblem will have no perturbed eigensolutions near the eigenspace of the unperturbed problem.

Degenerate roots

Degenerate multiple roots can lead to an expansion in non-integral powers of ϵ. Consider the n-degenerate eigensolution in the Jordan Normal Form

$$\begin{array}{lcll} A\mathbf{e}_1 & = & a\mathbf{e}_1 & \\ A\mathbf{e}_2 & = & a\mathbf{e}_2 & + \quad c_2\mathbf{e}_1 \\ \vdots & & \vdots & \vdots \\ A\mathbf{e}_n & = & a\mathbf{e}_n & + \quad c_n\mathbf{e}_{n-1} \end{array}$$

Then if the perturbation $\epsilon\mathbf{B}(\mathbf{e}_1)$ has a component in the direction $\mathbf{e}_n$, say ϵB_n, an expansion is needed in powers of $\epsilon^{1/n}$, i.e.

$$\begin{array}{lcll} \mathbf{x}(\epsilon) & = & \mathbf{e}_1 & + \quad \epsilon^{1/n}x_2\mathbf{e}_2 \quad + \epsilon^{2/n}x_3\mathbf{e}_3 + \cdots + \epsilon^{(n-1)/n}x_n\mathbf{e}_n + \cdots \\ \lambda(\epsilon) & = & a & + \quad \epsilon^{1/n}\lambda_1 \quad + \cdots \end{array}$$

with solution

$$\begin{aligned} x_2 = \lambda_1/c_2, \quad x_3 = \lambda_1^2/c_2c_3, \quad \cdots \quad x_n = \lambda_1^{n-1}/c_2c_3\ldots c_n \\ \text{and} \quad \lambda_1 = (c_2c_3\ldots c_nB_n)^{1/n} \end{aligned}$$

If the components of $\epsilon\mathbf{B}(\mathbf{e}_1)$ vanish in the directions of $\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \ldots, \mathbf{e}_n$ then an expansion is needed in powers of $\epsilon^{1/k}$.

Exercise 1.7. Find the second order perturbations of the eigenvalues of the matrix

$$\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

for small ω and for large ω. Consider to first order the 3×3 version of this problem.

Exercise 1.8. Find the first order perturbations of the eigenvalues of the differential equation

$$y'' + \lambda y + \epsilon y^n = 0$$

in $0 < x < \pi$, with $y(0) = y(\pi) = 0$ for $n = 1$, 2 and 3.